ARTICLE

Pseudo Folding Back When Students Solve Real Analysis Problems

Nusantara Journal of Social Sciences and Humanities
Issue 1: 2020
https://lekantara.com/index.php/njsh

Edi Purwanto

Universitas Pawyatan Daha Kediri
Kediri, Indonesia

Email: edi91purwanto@gmail.com

Makmun Solehudin

Universitas Pawyatan Daha Kediri
Kediri, Indonesia

Email: solehudinmakmun@gmail.com

Abstract
Folding back is an important element of Pirie-Kieren's theory of growth in mathematical understanding. Students do folding back to the inner level of understanding, when facing problems at a certain level of understanding. The main focus of the folding back is the thicker understanding of the deeper components. Susiswo (2015) explains that when students face problems at a certain level of understanding, they will return to the inner level of understanding, but the understanding does not become thicker. This condition is known as pseudo folding back. This study aims to describe students' pseudo folding back when solving Real Analysis problems.

Keywords: pseudo folding back, real analysis problem

Introduction
Understanding is an important element in learning, including learning mathematics. Students will not be successful in understanding a mathematical concept if the prerequisite concepts are not well understood. In order to do well in learning mathematics, students must understand the relationship of a concept to one before it. Lithner (2012) explains that in mathematics education, students are expected to be able to understand and solve math-

related problems, even after 30 years of research and reform, there are still many students who think efficiently by rote. Tall (2002) also explained that a common difficulty experienced by students in learning mathematics is that they think that mathematics studies are too abstract.

The main key of Piere and Kieren's (1994) theory of understanding is the folding back. When faced with a question or question at a certain level that cannot be resolved immediately, students need to fold back to a deeper layer to explore insufficient understanding at that time. The understanding of the deeper layers after the folding back occurs is not identical with the understanding of the original layer. The understanding that is formed at the deeper layer is influenced by the need to be able to solve problems at the outer layer. Students now have an understanding that is thicker at a deeper layer.

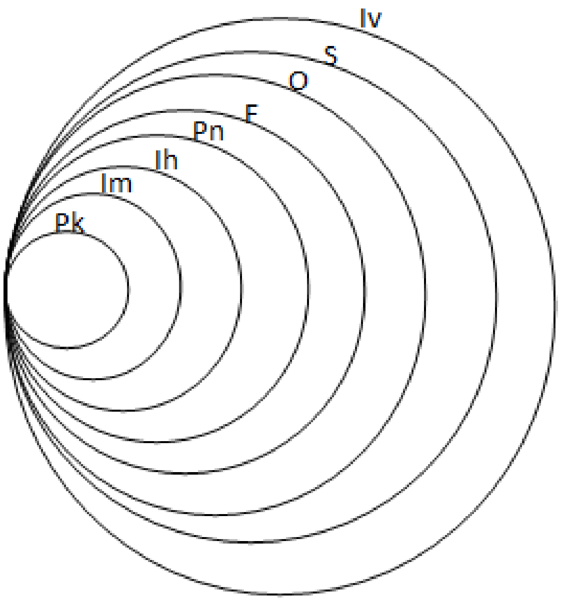
In understanding a mathematical concept, the occurrence of folding back between one student and another can be different. Certain students need to do folding back continuously to get a broader and deeper understanding. Meanwhile, other students do not need to do folding back repeatedly to be able to develop a broader and deeper understanding. This will result in each student having a different trajectory.

The key word for folding back is the understanding that is thicker in the inner layer. Susiswo (2015) explains that when facing problems at the understanding level of structuring, students return to the primitive knowing level of understanding but their understanding does not become thicker. For example, when asked to explain that there is no rational number r so that $r^2 = 2$, at the beginning students make the supposition that m and n have no common factors other than 1, through contradiction evidence the student wants to show that m and n are even numbers. At the end of the proof, students state that m^2 and n^2 are even numbers, but students do not realize that it is necessary to show m and n are even, so that they have a common factor other than 1 to contradict the presupposition in the beginning so that it can show the correct proof. Students cannot explain the relationship between the solution and the concept of contradiction. Students return from the understanding level of structuring to the primitive knowing level of understanding, but their understanding does not become thicker. This process is known as pseudo folding back. This study aims to describe students' pseudo folding back when solving Real Analysis problems.

Literature Review

Piere and Kieren (1994) explain that understanding is a growth process that is complete, dynamic, layered but not linear, and sustainable. They also categorize eight levels or layers of understanding, namely: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventising (Figure 1).

Figure 1. The Pirie-Kieren Model of Mathematical Comprehension Growth Level (Piere and Kieren, 1994)



Information:
Pk: primitive knowing,
Im: image making,
Ih: image having,
Pn: property noticing,
F: formalising,
O: observing,
S: structuring, and
Iv: inventising

Primitive Knowing

The first level of understanding is primitive knowing. Primitive knowing is the initial level of effort in understanding new definitions, using previous knowledge to the next level of understanding, through actions involving definitions, or representing definitions (Pirie and Kieren, 1994; Parameswaran, 2010; Manu, 2005). The understanding level of primitive knowing functions as material for constructing the next understanding (Droujkova et al, 2005; Warner and Schorr, 2004). According to Meel (2003), various names are related to the primitive knowing level of understanding, namely: "intuitive knowledge" by Leinhardt, "situated knowledge by Brown, Collins, and Duguid.

Image Making

The second level of understanding is called image making. Image making is a level where an understanding of prior knowledge is formed and uses it in new ways (Pirie and Kieren, 1994). In this case, students make a picture based on previous knowledge. Students try to understand a concept, either mentally or physically, to be able to get an idea about the concept. As a consequence, action in this layer involves developing a relationship between images and symbols (Meel, 2003). Students develop certain ideas and make a picture of a concept through pictures or through examples (Martin, et al., 2005).

Image Having

The third level of understanding is image having. In Image having students already have a description of a concept and make a mental picture of the concept (Pirie and Kieren, 1994), without having to work on examples. Concept images resulting from the previous level of understanding are replaced by mental images (Meel, 2003). So that this level is the first level of abstraction from students.

Property Noticing

The fourth level of understanding is property noticing. Students are able to combine aspects of a concept to form properties that are relevant and specific to the concept (Pirie and Kieren, 1994). The difference between image having and property noticing is the ability to understand the existence of a relationship between images of a concept and explain how to verify that relationship (Meel, 2003). Students realize the similarities and differences in the various descriptions of a concept and develop them into a definition that is built between the relationships of these images (Tall and Vinner, 1981).

Formalizing

The fifth level of understanding is formalizing. At this level students make an abstraction of a mathematical concept based on the properties that appear (Pirie and Kieren, 1994). So that at this level students are able to understand a definition or formal algorithm of mathematical concepts. The definition of a mathematical concept emerges as an entity that is free from the context that forms it (Droujkova et al, 2005).

Observing

The sixth level of understanding is observing. At this level students coordinate formal activities at the formalizing level so that they are able to use them on the related problems they face (Pirie and Kieren, 1994). So that students are able to link understanding mathematical concepts with new knowledge structures. Furthermore, students are able to make formal statements about a mathematical concept and are able to look for a pattern to determine the proof of a theorem.

Structuring

The seventh level of understanding is structuring. At this level students are able to link the relationship between one theorem and another and are able to prove it based on logical arguments (Pirie and Kieren, 1994). So that students are able to prove the relationship between one theorem and another axiomatically.

Inventising

The final level of understanding is inventising. At this level students have a complete structured understanding and create new questions that can grow into a new concept (Pirie and Kieren, 1994). So that students are able to create new mathematical structures based on previous knowledge structures. Students' mathematical understanding is not limited and goes beyond the existing structure so that they are able to answer the question "what if?" (Meel, 2003).

Research Method

This research was conducted in the Mathematics Education Study Program of FIP, Pawyatan Daha University, Kediri. Because this research is to describe pseudo folding back in solving Real Analysis problems, the selected subjects are students who have taken the Real Analysis course.

The research subject was taken based on the existence of pseudo folding back in students when solving problems about real analysis. At first the subject was asked to solve real analysis problems. Based on this settlement, interviews were conducted to determine the folding back. The subject selection process is as follows:

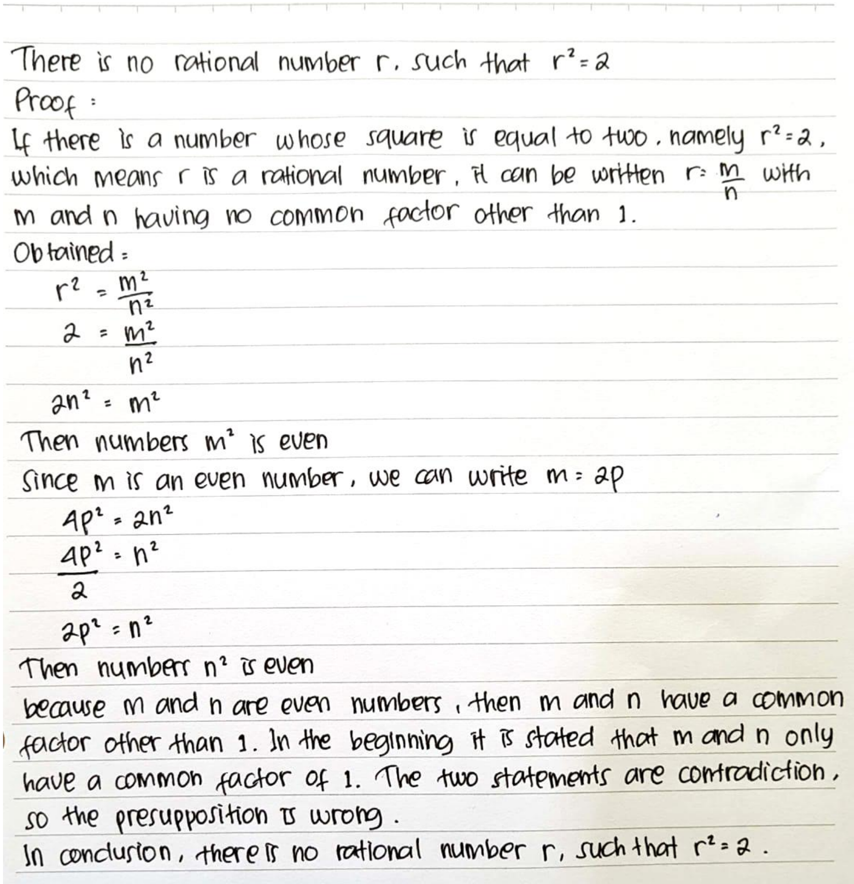
1. Classical tests were conducted on students in semester 4. Students in semester 4 were chosen because these students had taken the real analysis course.
2. Test questions contain questions about the nature of algebra, rational numbers, irrational numbers, and limit. This material was chosen because it is the main prerequisite material for real numbers.
3. The test is used to group students into 2 groups, namely a group of prospective subjects who have a tendency to conceptual knowledge and a group of prospective subjects who have procedural tendencies. Conceptual and procedural knowledge in this study adopted in Hibert (1986), Skemp (1987), and Kilpatrick, et al. (2001).
4. Each group is taken by one student as a prospective research subject.
5. The selected subject is the subject with the "best" solution in terms of the above grouping.
6. Prospective research subjects solve real analysis problems.
7. Interviews were conducted to determine the subject's pseudo folding back in solving real analysis problems.
8. If pseudo folding back does not occur, then the student is not the subject of research.
9. Return to step 5.
10. Subjects from the group that tend to conceptual and procedural knowledge are referred to as Subject 1 and Subject 2, respectively.

Data collection was carried out through tests and interviews. The test was conducted as a reference in the interview, to find out the subject's pseudo folding back. After obtaining the research subject, in this case Subject 1 and Subject 2 were obtained, then each subject was asked to solve the following real analysis problems.

"Prove that there is no rational number r, such that $r^2 = 2$!"

Problem solving from two subjects as in Figure 2.

Figure 2. Problem Solving (The Result of Student Answers, 2020)


There is no rational number r, such that $r^2 = 2$
Proof :
If there is a number whose square is equal to two, namely $r^2 = 2$, which means r is a rational number, it can be written $r = \frac{m}{n}$ with m and n having no common factor other than 1.
Obtained :
$r^2 = \frac{m^2}{n^2}$
$2 = \frac{m^2}{n^2}$
$2n^2 = m^2$
Then numbers m^2 is even
Since m is an even number, we can write $m = 2p$
$4p^2 = 2n^2$
$\frac{4p^2}{2} = n^2$
$2p^2 = n^2$
Then numbers n^2 is even
because m and n are even numbers, then m and n have a common factor other than 1. In the beginning it is stated that m and n only have a common factor of 1. The two statements are contradiction, so the presupposition is wrong.
In conclusion, there is no rational number r, such that $r^2 = 2$.

two subjects can solve real analysis problems as shown in Figure 1, so that the two students are at the level of understanding structuring. Based on the solution above, an interview was conducted to describe the pseudo folding back of the two subjects. The two subjects were asked to relate their solution to the even number concept.

Results and Discussion

Pseudo folding back occurs when students face problems that cannot be resolved immediately at a certain level which requires students to fold back to the previous level of understanding, but there is no thicker understanding at the previous level. Pseudo folding back will be described in advance. Subject 1.

Subject 1 can solve the problem, prove that there is no rational number *r*, such that $r^2 = 2$!

As seen in Figure 1, Subject 1 stated that the solution was obtained with evidence of contradiction. Assuming that there is a number *r* squared the result is two written $r^2 = 2$, because r is assumed to be a rational number, it is written for $r=\frac{m}{n}$ where *m* and *n* have no common factor other than 1. Pseudo folding back occurs when Subject 1 is asked to explain the statement that obtained $r^2=\frac{m^2}{n^2}$, then $2=\frac{m^2}{n^2}$, and obtained $2n^2=m^2$, subject 1 states that m^2 is an even number, consequently *m* is also an even number. Because it is an even number, write $m = 2p$ (Interview 1). However, subject 1 does not provide a condition that $p \epsilon \mathbb{Z}$. As explained by Sidebotham (2003) that the set of even numbers = {2k: kϵ $\mathbb{Z}$ }. Subject 1 is incomplete in providing the conditions for m and n, the complete condition is that it does not have a common factor other than 1, *m* and *n* are integer elements, and $n \neq 0$. Subject 1 folds back to the level of understanding primitive knowing about the concept of even numbers to conclude that $m^2$is an even number, but does not show any proof that m will have an even effect, so that his understanding does not become thicker.

***Researcher**: "Try to explain this conception!"*
***Subject 1**: "Initially it is assumed that if there is a number r squared the result is two, written $r^2=2$, then because r is considered rational, it is written $r=\frac{m}{n}$, with m and n having no common factor other than 1. So because $\frac{m}{n}$ is just squared the same way $2= r^2= \frac{m^2}{n^2}$, we get $2= \frac{m^2}{n^2}$, then this is $2n^2=m^2$ because it's multiplied by cross, sir. So m^2 is an even number because it is equal to $2n^2$. As a result m is also an even number, so m = 2p "*

Interview 1 Subject 1 shows the first pseudo folding back

The second pseudo folding back occurs when Subject 1 is asked to explain that m^2 is an even number, consequently m is also an even number. Since it is an even number, write m = 2p. Furthermore, because for $m^2=2n^2$, so that $4p^2=2n^2$ is obtained, $2p^2=n^2$ is obtained. Furthermore, it is stated that n^2 is an even number, consequently n is also an even number (interview 2). Subject 1 folds back to the level of understanding primitive knowing about the concept of even numbers to conclude that n^2 is an even number, but does not show evidence that *n* will have an even effect, so that his understanding does not become thicker.

***Researcher**: "then how do you show a contradiction?"*
***Subject 1**: "because earlier m^2 is an even number so consequently m is also an even number. Since it is an even number, write m = 2p. Then because this is for $m^2=2n^2$, so this is 2p squared first to get $4p^2=2n^2$, then we get $2p^2=n^2$. so it is stated that n2 is an even number, consequently n is also an even number. Hehe.. (smile)*

Interview 2 Subject 1 shows pseudo folding back

Furthermore, the pseudo folding back will be described in Subject 2.The first pseudo folding back occurs when Subject 2 explains the reasons for completion,

$$\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) =$$
$$\lim_{x\to 1}\left(\frac{(x+1)(x-1)}{x-1}\right) =$$
$$\lim_{x\to 1}(x+1) = 2$$

Subject 2 said that dividing *x - 1* by *x - 1* results in 1 (Interview 3). Subject 2 folds back to the primitive knowing level of understanding. Subject 2 performed a division procedure involving the variable *x* but did not see the connection between the solution and the limit concept so that the understanding of the limit concept at the primitive knowing level did not become thicker.

***Researcher**: "what is the reason for getting this first row to second row?"*
***Subject 2**: "This Limits x approaches one, extract x plus one times x minus one by x minus one, this means that limit x approaches one, then what x minus one is divided by x minus one is equal. So just limit x approaches one for x plus one, the result is 2. "*

Interview 3 Subject 2 Saying that there is a division of *x - 1* by *x - 1* the result is 1

The second pseudo folding back occurs still at completion,

$$\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) = \lim_{x\to 1}\left(\frac{(x+1)(x-1)}{x-1}\right) = \lim_{x\to 1}(x+1) = 2$$

Subject 2 states that the solution is obtained from two functions $f(x) = \frac{(x+1)(x-1)}{x-1}$ and $g(x) = x + 1$which are the same two functions (Interview 4). Subject 2's work is not linked to a formal definition of limit.

> *To say* $\lim_{x\to c} f(x) = L$*means that for is given* $\varepsilon > 0$ (*no matter how small*) *there is are corresponding* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon$, *provided that* $0 < |x - c| < \delta$, *that is,* $0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$(Bartle and Sherbert, 2011).

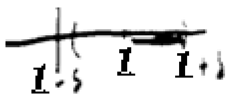
Subject 2 folds back to the primitive knowing level of understanding about the concept of the similarity of two functions but his understanding is wrong. Thus, the understanding of Subject 2 on this primitive knowing level does not become a thicker.

***Researcher**: "stated that this is a function, first line f (x), second line g (x). Are the values of f (x) and g (x) the same value? "* ***Subject 2**: "Yes, Sir."*

Interview 4 Subject 2 states that the two functions *f (x)* and *g (x)* are two of the same Functions

The third pseudo folding back occurs when Subject 2 is asked to associate the statement *x* close to one but not equal to one with the formal definition of limit. Subject 2 states that the statement is derived from the inaccuracy $| x - 1 | < \delta$. Subject 2 drew the solution of the inequality (Figure 3), then explained the areas $1 - \delta$ and $1 + \delta$ (Interview 5). Subject 2 folds back to the primitive knowing level of understanding. Subject 2 worked on the concept of inequality of absolute values but was wrong in ignoring it with the statement *x* approaching one but not equal to one. Comprehension of Subject 2 on the primitive knowing level does not become a thicker.

Figure 3. Graph of Inequality Resolution | x - 1 | <δ (The Result of Student Answers, 2020)



***Researcher**:" then which other statement x approaches which one of the statements?"* ***Subject 3**: "x minus this one is less than delta."* ***Researcher**: "try to tell me how it can be like that."* ***Subject 3**: "This is 1, enter the delta, Sir. One plus delta, if here means one plus delta minus one is equal to delta."*

Interview 5 Explaining Inequality | x - 1 | <δ

The fourth pseudo folding back occurs when Subject 2 is asked to compile formal proof of limits. The subject stated that because the absolute value of x minus one is less than delta, the delta is taken equal to epsilon (Interview 6). Subject 2 misunderstood the meaning for each epsilon there is a delta. Supposedly, Subject 2 takes any $\varepsilon > 0$ first then chooses $\delta = \varepsilon$. Therefore, Subject 1's understanding at the primitive knowing level does not become a thicker.

***Researcher**: "What is the basic analysis process using? Try to explain!"*
***Subject 2**: "Refer to the definition definition sir. Since the absolute value of x minus one is less than delta, the delta is taken to be equal to epsilon. "*

Interview 6 Analysis for Taking Delta

Discussion

The results showed that there were three main things that caused the subject to do pseudo folding back, namely first when the subject explained the statement that $r^2=\frac{m^2}{n^2}$was obtained, then $2=\frac{m^2}{n^2}$, and obtained $2n^2=m^2$, based on the definition it is stated that m^2 is an even number, consequently m is also an even number. Second, when the subject explains the statement x approaches one but is not equal to one. And third, when the subject explains the statement for each $\varepsilon> 0$ there is $\delta> 0$. These three problems are also the main problems in early genetic decomposition by Cotril, et al. (1996).

In the first problem, subjects who have a tendency towards conceptual knowledge do not do pseudo folding back, while subjects who have a tendency towards procedural knowledge fold back to the definition of even numbers. This is in line with the results of research conducted by Duru (2011), which states that subjects tend to experience misconceptions. Duru (2011) also explained that when solving problems, the subject tends to be procedural skills. Subjects who make mistakes like this are called pseudo analytical behavior (Vinner, 1997).

For the second problem, the two subjects performed pseudo folding back. Subjects who have a tendency to conceptual knowledge fold back to the definition of absolute value, while subjects who have a tendency towards procedural knowledge fold back to the absolute value inequality solving procedure. The first subject did pseudo conceptual while the second subject did pseudo analytical (Vinner, 1997). Ciltas & Tatar (2011) explained that the subject had difficulty solving the problem of inaccuracy of absolute values and had difficulty interpreting the intervals found.

for the third problem, both subjects performed pseudo folding back. Both subjects both do not understand the statement for each $\varepsilon> 0$ there is $\delta> 0$. Juter (2007) states that the cognitive problem that arises in understanding the definition of limit is when understanding the quantity in the formal definition of limit and making the link between the formal definition of limit and the problem of limit. One of the sources of student difficulty is understanding the conventional definition of limit $\varepsilon - \delta$ (Cotril, et al., 1996). Misunderstanding is the result of a poor understanding of the basic theme of function limits and representations (Muzangwa & Chifamba, 2012).

Conclusion

Logical and systematic proof skills are the basis for students to find a basis for theory, besides that it needs to be supported by directed fold backs. The formal definition of limit is the basis for students to continue to study mathematics which is more formal and more stringent (Swinyard & Larsen, 2012). The results showed that there were three main things that caused the subject to do pseudo folding back first when the subject explained the statement that $r^2=\frac{m^2}{n^2}$ was obtained, then $2=\frac{m^2}{n^2}$, and obtained $2n^2=m^2$, based on the definition it is stated that m^2 is an even number, consequently m is also an even number. Second, when the subject explains the statement x approaches one but is not equal to one.

And third, when the subject explains the statement for each $\varepsilon > 0$ there is $\delta > 0$. Suggestion for learning is that the prerequisite material must be fully understood before discussing real analysis.

Limitation of the Study

In this study, researchers only limited it to real analysis studies and focused on the occurrence of pseudo folding back based on Piere and Kieren's (1994) theory.

Declaration of conflicting interest

In writing this article, the author has obtained permission from the agency where the research subject comes from. so that there will be no conflicts of interest between authors and others.

Funding

In preparing this research article, the author uses personal funds. The purpose of this article is as an assignment to doing the tridharma of college.

References

1. Bartle, Robert G. & Sherbert, Donald R. (2011) *Introduction to real analysis*. Urbana: John Wiley & Sons, Inc.
2. Ciltas, A. & Tatar, E. (2011) diagnosing learning difficulties related to the equation and inequality that contain terms with absolute value. *International Online Journal of Educational Sciences,* Volume 3, Nomor 2: 461-473.
3. Cottrill, J., Dubinsky, Ed, & Nichols, D. (1996) Understanding the limit concept: beginning with a coordinated process schema. *Journal of Mathematical Behavior*. Vol.15: 167-192. http://dx.doi.org/10.1016/S0732-3123(96)90015-2
4. Droujkova A. M., Berenson B. S., Slaten, K., & Tombes S. (2005) A conceptual framework for studying teacher preparation: The pirie-kieren model, collective understanding, and metaphor. *Proceedings of the 29th conference of the international group for the psychology of mathematics education*, Vol. 2, pp. 289-296.
5. Duru, A. (2011) Pre-Service teachers' perception about the concept of limit. *Kuram ve Uygulamada Eğitim Bilimleri Educational Sciences: Theory and Practice*. Vol.11, Numbers 3: 1710-1715.
6. Hiebert, J. (1986) *Conceptual and prosedural knowledge: The case of mathematics.* London: Lawrence Erlbaum Associates.
7. Juter, K. (2007) Students' conceptions of limits: High achievers versus low achievers*. The Montana Mathematics Enthusiast*, ISSN 1551-3440, Vol. 4, no.1, pp. 53-65.
8. Kilpatrick, J., Swafford, J., dan Findell, B. (2001) *Ading It Up Helping Children Learn Mathematics*. Washington DC. National Research Council.
9. Lithner, Johan. (2012) Learning mathematics by creative or imitative reasoning. *International Congress on Mathematical Education*.
10. Martin, C., LaCroix, L. & Fownes, L. (2005) Folding back and the growth of mathematical understanding in workplace training. *Adults Learning Mathematics An International Journal.*Vol.1, Nomor 1.
11. Meel, E. (2003) Model and theories of mathematical understanding: Comparing pirie- kieren's model of the growth of mathematical understanding and APOS theory. *CBMS Issues in Mathematics Education.*Vol. 12.
12. Muzangwa, J. dan Chifamba, P. (2012) Analysis of errors and misconceptions in the learning of calculus by undergraduate students. *Acta didactica napocensia.* Volume 5 Nomer 2.Parameswaran, R. (2010). Expert mathematicians' approach to understanding definitions. *The mathematics educator*. Volume 20, Nomor 1: 43-51.
13. Pirie, S. & Kieren, T. (1994) Growth in mathematical understanding: How we can characterize it and how we can represent it. *Educational studies in mathematics*, Vol. 9: 160–190. http://dx.doi.org/10.1007/978-94-017-2057-1_3.
14. Sidebotham, Thomas H. (2003) *The a to z of mathematics: a basic guide,* Hlm. 181, ISBN 9780471461630

15. Skemp, R. (1987) *Psychology of learning mathematics.* Lawrence erlbaum associates. New Jersey. Hillsdale.
16. Susiswo (2015) Students' form folding back in solving limit problems. *Disertasi doktor pascasarjana.* universitas negeri malang, pendidikan matematika.
17. Swinyard, C. & Larsen, S. (2012) Coming to understand the formal definition of limit: insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education.* Vol. 43, No. 4, 465-493.
18. Tall, D. (2002) *Advanced mathematical thinking*. New York : Kluwer Academic Publisher.
19. Tall, D.& Vinner, S. (1981) Concept image and concept definition in mathematics with particular reference to limits and continuity. *educational studies in mathematics*. Vol.12: 151–169. http://dx.doi.org/10.1007/BF00305619.
20. Vinner, S. (1997) The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational studies in mathematics*. Volume 34: 97-129. 69–302. https://doi.org/10.1628/001522112x653840